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Note

Injective Choice Functions

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The present paper is concerned with a combinatorial question called the "marriage problem." A criterion will be proved for the existence of an injective choice function of families with at most finitely many infinite members and a generalization of a theorem of H. A. Jung and R. Rado. We give a new proof of a theorem of J. Folkman.

Let $F = (F(i) : i \in I)$ be a family of nonempty sets $F(i)$ with the index set I. We call a function f a *choice function of* F if the domain of f is I and $f(i)$ is an element of $F(i)$, for all elements i of I. Since there is no injective choice function of the family $F = \{(0, \{0, 1\}), (1, \{0\}), (2, \{1\})\}$, we are interested in a criterion (necessary and sufficient) for the existence of an injective choice function of a given family. We shall use the letters F, G, H for families of nonempty sets, the letters I, J, K for index sets, and the letters f, g, h for choice functions. We write $\operatorname{rng} f$ for the range of f, $\operatorname{dom} f$ for the domain of f, and $f \restriction K$ $(F \restriction K)$ for the restriction of $f(F)$ to the set K. $P(I)$ denotes the power set of I, $P_\omega(I)$ the set of all finite subsets of I, and $\omega = \{0, 1, 2, 3, \ldots\}$, the set of all natural numbers. The letter n always denotes a natural number. $IA(F, f)$ means that f is an injective choice function of the family F. Let us call a family F *critical if* (1) $\exists f\, IA(F, f)$ and (2) $\neg \exists f\, (IA(F, f)\, \&\, \operatorname{rng} f \subsetneqq \bigcup \operatorname{rng} F)$.

LEMMA 1. *Let $F = (F(i) : i \in I)$ be a family. Then*

$$\exists f\, IA(F, f) \Rightarrow \neg \exists K \subseteq I\, \exists i \in I \backslash K \left(F(i) \subseteq \bigcup \operatorname{rng} F \restriction K \,\&\, F \restriction K \text{ critical}\right).$$

The proof is straightforward. We hope that the necessary condition of Lemma 1 is also sufficient. First we attack the problem for finite families.

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THEOREM 1 (P. Hall). *Let $F = (F(i) : i \in n)$ be a finite family. Then*

$$\exists f\, IA(F, f) \Leftrightarrow \forall J \subseteq n \left(\left|\bigcup_{j \in J} F(j)\right| \geqslant |\, J \,|\right).$$

THEOREM 2. *Let $F = (F(i) : i \in n)$ be a finite family. Then*

$$\exists f\, IA(F, f) \Leftrightarrow \neg\exists K \subseteq n\, \exists i \in n \backslash K \left(F(i) \subseteq \bigcup \operatorname{rng} F \upharpoonright K \;\&\; F \upharpoonright K \textit{ critical}\right).$$

Proof. The one direction of Theorem 2 is Lemma 1. The other direction follows by an argument of the referee. Suppose F did not have an injective choice function. Then by P. Hall's theorem, there exists $K \subseteq n$ such that $|\bigcup_{i \in K} F(i)| < |\, K \,|$. Choose K minimal with respect to this property. Thus $|\bigcup \{F(i) : i \in K\backslash\{j\}\}| \geqslant |\, K\backslash\{j\}| = |\, K \,| - 1$ for all $j \in K$. Therefore $|\, K \,| -1 \leqslant |\bigcup \{F(i) : i \in K\backslash\{j\}\}| \leqslant |\bigcup \{F(i) : i \in K\}| \leqslant |\, K \,| -1$, so that $\bigcup \{F(i) : i \in K\backslash\{j\}\} = \bigcup \{F(i) : i \in K\}$ and $F(j) \subseteq \bigcup \{F(i) : i \in K\backslash\{j\}\}$ for all $j \in K$. But by minimality of K, $|\bigcup_{i \in I} F(i)| \geqslant |\, I \,|$ for all $I \subseteq K\backslash\{j\}$ so that $(F(i) : i \in K\backslash\{j\})$ is critical by Theorem 1 with $F(j) \subseteq \bigcup \{F(i) : i \in K\backslash\{j\}\}$ for all $j \in K$.

It is not difficult to prove:

LEMMA 2. *Let $F = (F(i) : i \in I)$ be a family which has an injective choice function* (i.c.f.), *and let $\mathfrak{K}$ be a subset of $P(I)$. If $F \upharpoonright K$ is critical for all $K \in \mathfrak{K}$, then $(F(k) : k \in \bigcup \mathfrak{K})$ is a critical subfamily of F.*

LEMMA 3. *Let $F = (F(i) : i \in I)$ be a family, let $K \subseteq I$, $i_0 \in I\backslash K$, $F \upharpoonright K$ critical, $F(i_0) \subseteq \bigcup \operatorname{rng} F \upharpoonright K$, and let $k_0 \in K$. Then*

$$IA(F \upharpoonright (K\backslash\{k_0\}) \cup \{i_0\}, f) \Rightarrow \operatorname{rng} f = \bigcup \operatorname{rng} F \upharpoonright K.$$

Proof. We use a zick-zack argument. Let f be an i.c.f. of

$$F \upharpoonright (K\backslash\{k_0\}) \cup \{i_0\}$$

and g an i.c.f. of $F \upharpoonright K$. We define a zick-zack (x_n) by induction. Let $x_0 = i_0$. If x_n has been defined, we let x_{n+1} be undefined, if $x_n = k_0$. The case $x_n \neq k_0$ implies $x_n \in \operatorname{dom} f$, and since $F \upharpoonright K$ is critical, $f(x_n)$ is an element of $\operatorname{rng} g$. Therefore we define $x_{n+1} = g^{-1}(f(x_n))$, if $x_n \neq k_0$. The zick-zack (x_n) is a finite sequence. Assume that (x_n) is not finite. Then $(g\backslash\{(x_n, g(x_n)) : 1 \leqslant n < \omega\}) \cup \{(x_n, f(x_n)) : n < \omega\}$ is an i.c.f. of $F \upharpoonright K \cup \{i_0\}$. Contradiction! Let $x_{n_0} = k_0$. The function

$$h = (f\backslash\{(x_n, f(x_n)) : 0 \leqslant n < n_0\}) \cup \{(x_n, g(x_n)) : 1 \leqslant n \leqslant n_0\}$$

is an i.c.f. of $F \upharpoonright K$ and $\operatorname{rng} f = \operatorname{rng} h = \bigcup \operatorname{rng} F \upharpoonright K$.

COROLLARY. Let $F = (F(i): i \in I)$ *be a family,* $K \subseteq I$, $i_0 \in I \backslash K$, $F \restriction K$ *critical,* $F(i_0) \subseteq \bigcup \operatorname{rng} F \restriction K$, *and let* $k_0 \in K$. *If there is any* i.c.f. *of*

$$F \restriction (K \backslash \{k_0\}) \cup \{i_0\},$$

then $F \restriction (K \backslash \{k_0\}) \cup \{i_0\}$ *is a critical family.*

We call a family with only finite members a *Hall-family.*

THEOREM 3 (M. Hall). *A Hall-family F possesses an* i.c.f. *iff every finite subfamily of F possesses an* i.c.f.

If $F = (F(i): i \in \omega)$ is a countable Hall-family, and if we define $f \leqslant g$ iff $f \subseteq g$ for $f, g \in \{h: \exists n \in \omega\ IA(F \restriction n, h)\}$, then M. Hall's result is an easy consequence of König's infinity lemma. You find a proof of Theorem 3 in the excellently written book of L. Mirsky.

THEOREM 4. *Let* $F = (F(i): i \in I)$ *be a family, let* $n \subseteq I$, *and let* $F \restriction I \backslash n$ *be a Hall-family. Then*

$$\exists f\, IA(F, f) \Leftrightarrow \neg \exists K \subseteq I\, \exists i \in I \backslash K\, (F(i) \subseteq U \operatorname{rng} F \restriction K \,\&\, F \restriction K \text{ critical}).$$

Proof. By induction. For $n = 0$, the assertion follows from Theorems 2 and 3. Suppose that the assertion holds for n. Let $F = (F(i): i \in I)$ be a family, let $n + 1 \subseteq I$ and $F \restriction I \backslash n + 1$ be a Hall-family, and let F satisfy the condition of Theorem 4. By the induction hypothesis, $F \restriction I \backslash \{n\}$ has an i.c.f. Let us assume that F has no i.c.f. and let $x \in F(n)$.

Case 1. $\exists i \in I \backslash \{n\}\, F(i) = \{x\}$. Let i_0 be such an i. Then $H = \{(i_0, F(i_0))\}$ is a critical subfamily and $x \in \bigcup \operatorname{rng} H$.

Case 2. $\forall i \in I \backslash \{n\}\ F(i) \neq \{x\}$. Let $F^*(i) = F(i) \backslash \{x\}$ for $i \in I \backslash \{n\}$. The family $F^* = (F^*(i): i \in I \backslash \{n\})$ has no i.c.f. By induction hypothesis, there exists an element $i_0 \in I \backslash \{n\}$ and a subset $K \subseteq I \backslash \{n\}$ such that $i_0 \notin K$, $F^* \restriction K$ is critical and $F^*(i_0) \subseteq \bigcup \operatorname{rng} F^* \restriction K$. We shall prove that $(F(j): j \in K \cup \{i_0\})$ is critical. $F \restriction K \cup \{i_0\}$ has an i.c.f. since

$$F \restriction K \cup \{i_0\} \subseteq F \restriction I \backslash \{n\}.$$

Assume that f is an i.c.f. of $F \restriction K \cup \{i_0\}$, and let $y \in (\bigcup \operatorname{rng} F \restriction K \cup \{i_0\}) \backslash \operatorname{rng} f$. It is $x \neq y$, since $F^* \restriction K \cup \{i_0\}$ possesses no i.c.f. Assume $f(i_0) = x$. If $y \in F^*(i_0)$, then $F^* \restriction K \cup \{i_0\}$ has an i.c.f. If $y \notin F^*(i_0)$, then $F^* \restriction K$ is not critical. In every case there is a contradiction and therefore $f(i_0) \neq x$. Let $k_0 \in K$ and $f(k_0) = x$. Since $f^* = f \restriction (K \backslash \{k_0\}) \cup \{i_0\}$ is an i.c.f. of

$$F^* \restriction (K \backslash \{k_0\}) \cup \{i_0\},$$

it follows by Lemma 3 $\operatorname{rng} f = \bigcup \operatorname{rng} F^* \restriction K$. $x \neq y$ implies

$$y \in \bigcup \operatorname{rng} F^* \restriction K \cup \{i_0\}.$$

Otherwise there is $\bigcup \operatorname{rng} F^* \restriction K \cup \{i_0\} = \operatorname{rng} F^* \restriction K$. Therefore y is an element of $\bigcup \operatorname{rng} F^* \restriction K = \operatorname{rng} f^*$. Contradiction! We have proved that $F \restriction K \cup \{i_0\}$ is a critical family. Hence x is an element of $\bigcup \operatorname{rng} F \restriction K \cup \{i_0\}$.

Each $x \in F(n)$ and each case leads to a critical subfamily $F \restriction K_x$ of $F \restriction I \backslash \{n\}$. By Lemma 2 $\bigcup \{F \restriction K_x : x \in F(n)\}$ is a critical family and $F(n)$ is a subset of $\bigcup \operatorname{rng} \bigcup \{F \restriction K_x : x \in F(n)\}$, contrary to assumption. Hence F possesses an i.c.f.

Remark. This argument is also a proof of Theorem 2 without using Hall's theorem. Then Hall's theorem is a corollary of Theorem 2.

LEMMA 4. (Replacement Lemma). *Let $F = (F(i): i \in I)$ be a family, let $K \subseteq I$, $i_0 \in I \backslash K$, $F(i_0) \subseteq \bigcup \operatorname{rng} F \restriction K$ and $F \restriction K$ critical. Then there exists an element $k_0 \in K$ such that $F \restriction (K \backslash \{k_0\}) \cup \{i_0\}$ is critical.*

Proof. Case 1. $\exists k \in K \exists f\, IA(F \restriction (K \backslash \{k\}) \cup \{i_0\}, f)$. Then by the corollary of Lemma 3, there is nothing to prove. Case 2.

$$\neg \exists k \in K \exists f\, IA(F \restriction (K \backslash \{k\}) \cup \{i_0\}, f).$$

Let $(k_\alpha)_{\alpha<\beta}$ be a bijection of the ordinal β onto the set K. Assume that there exists an element $\alpha < \beta$ and an i.c.f. of $F \restriction K$ such that $f(k_\alpha)$ is an element of $F(i_0)$. Then $F \restriction (K \backslash \{k_\alpha\}) \cup \{i_0\}$ possesses an i.c.f. Contradiction! Hence $\{f(k_\alpha): IA(F \restriction K, f)\} \cap F(i_0) = \varnothing$ for all $\alpha < \beta$. This means that

$$\bigcup_{\alpha<\beta} \{f(k_\alpha): IA(F \restriction K, f)\} \cap F(i_0) = \varnothing.$$

Since $\bigcup_{\alpha<\beta} \{f(k_\alpha): IA(F \restriction K, f)\} = \bigcup \operatorname{rng} F \restriction K$, we get a contradiction to $\varnothing \neq F(i_0) \subseteq \bigcup \operatorname{rng} F \restriction K$. We have proved that case 2 is impossible.

THEOREM 5. *Let $F = (F(i): i \in I)$ be a family, let $n \subseteq I$, and let $F \restriction I \backslash n$ be a Hall-family. Then $\exists f\, IA(F, f) \Leftrightarrow \bigcup \{G : G \subseteq F \,\&\, G \text{ critical}\}$ critical.*

Proof. The direction "$\Rightarrow$" follows from Lemma 2. Suppose that F possesses no i.c.f. By Theorem 4 there is a set $K \subseteq I$ and an element $i_0 \in I \backslash K$ such that $F(i_0) \subseteq \bigcup \operatorname{rng} F \restriction K$ and $F \restriction K$ is critical. By the replacement lemma, there exists an element $k_0 \in K$ such that $F \restriction (K \backslash \{k_0\}) \cup \{i_0\}$ is a critical family. Since $(F \restriction (K \backslash \{k_0\}) \cup \{i_0\}) \cup F \restriction K = F \restriction K \cup \{i_0\}$ possesses no i.c.f., the family $\bigcup \{G \subseteq F : G \text{ critical}\}$ is not critical.

LEMMA 5 (Brualdi, Scrimger, Folkman, and Steffens). *If the Hall-family F possesses an* i.c.f., *then* $\bigcap\{\operatorname{rng} f : IA(F,f)\} = \bigcup\{\operatorname{rng} G : G \subseteq F$ *critical & G finite*$\}$.

Proof. Since the inclusion "$\supseteq$" is obvious, let $x_0 \in \bigcap\{\operatorname{rng} f : IA(F,f)\}$. If there is an element $i_0 \in I$ such that $F(i_0) = \{x_0\}$, then $x_0 \in \bigcap\{\operatorname{rng} G : G \subseteq F$ critical & G finite$\}$. Let $F(i) \neq \{x_0\}$ for all $i \in I$. The family

$$F^* = (F(i)\backslash\{x_0\} : i \in I)$$

possesses no i.c.f., and therefore there exists, by Theorem 3, a finite subfamily $(F^*(j) : j \in J)$ which possesses no i.c.f. A Hall-argument (Theorem 1) shows that $F \upharpoonright J$ is a critical family and x_0 is an element of $\bigcup \operatorname{rng} F \upharpoonright J$.

COROLLARY. *If F is a critical Hall-family, then* $F = \bigcup\{G \subseteq F : G$ *critical & G finite*$\}$.

Now there is no difficulty to extend the result of Lemma 5 to families with finitely many infinite members (without the finite property of G). As an application of Lemma 2, Theorem 4, Lemma 5, and the corollary to Lemma 5, we give a new proof of a theorem of J. Folkman.

THEOREM 6 (J. Folkman). *Let $F = (F(i) : i \in I)$ be a family, $n \subseteq I$, and $F \upharpoonright I\backslash n$ a Hall-family. Then* $\exists f\, IA(F,f) \Leftrightarrow \forall K \forall J \forall r \in \omega\, (K \subseteq J \subseteq I\ \&$ $\bigcup \operatorname{rng} F \upharpoonright J \backslash \bigcup \operatorname{rng} F \upharpoonright K$ *finite* & $|J\backslash K| \geqslant r + |\bigcup \operatorname{rng} F \upharpoonright J \backslash \bigcup \operatorname{rng} F \upharpoonright K| \Rightarrow$ $\exists S \in P_\omega(K)\, \forall T(S \subseteq T \subseteq K \Rightarrow |\bigcup \operatorname{rng} F \upharpoonright T| \geqslant r + |T|))$.

Proof. "$\Rightarrow$", look at [2]. We prove the other direction of Theorem 6 by induction. Let $n = 0$ and suppose that F has no i.c.f. By Theorem 3 there is a finite subset L of I such that $F \upharpoonright L$ possesses no i.c.f. By Theorem 4 there is a set $K \subseteq L$, an element $1_0 \in L\backslash K$ such that $F(1_0) \subseteq \bigcup \operatorname{rng} F \upharpoonright K$ and $|\bigcup \operatorname{rng} F \upharpoonright K| = |K|$. If $J = K \cup \{1_0\}$, then

$$|J\backslash K| = 1 + |\bigcup \operatorname{rng} F \upharpoonright J \backslash \bigcup \operatorname{rng} F \upharpoonright K| .$$

Let S be a subset of K. Since $|\bigcup \operatorname{rng} F \upharpoonright K| < 1 + |K|$, we choose for the set T the set K.

Now suppose that the assertion holds for every family with n many infinite sets. Let F be a family of $n + 1$ many infinite sets which satisfies the condition of Theorem 6. Assume that F possesses no i.c.f. By Lemma 2 $G = \bigcup\{H : H \subseteq F \upharpoonright I\backslash\{0\}$ critical$\}$ is a critical subfamily of F. Then $F(0) \subseteq \bigcap\{\operatorname{rng} f : IA(F \upharpoonright I\backslash\{0\}, f)\} = \bigcup\{\operatorname{rng} H : H \subseteq F \upharpoonright I\backslash\{0\}$ critical$\} =$ $\bigcup \operatorname{rng} G$. We note that $F(i) \in \operatorname{rng} G$ for all $i \in \{1, \ldots, n\}$, otherwise there is a contradiction by the induction hypothesis. Let f be an i.c.f. of G and define for $i \in \operatorname{dom} G\backslash\{1,\ldots,n\}$ $G^*(i) = F(i)\backslash\{f(j) : j \in \{1,\ldots,n\}\}$. The family G^* is a critical family of finite sets, since G is critical. By the corollary to Lemma 5, G^* is the union of all finite, critical subfamilies of G^*. Hence there exists to every finite subfamily H^* of G^* a finite, critical family K^* such that $H^* \subseteq K^* \subseteq G^*$. Let

$$r = \left|\{f(i): i \in \{1,\ldots,n\}\} \cap \bigcup \operatorname{rng} F \restriction \operatorname{dom} G^* \right|,$$

$$J = \operatorname{dom} G \cup \{0\}, \qquad \text{and} \qquad K = \operatorname{dom} G^*.$$

$| \bigcup \operatorname{rng} F \restriction J \backslash \bigcup \operatorname{rng} F \restriction K | = n - r$ implies

$$| J\backslash K | = n + 1 = (r + 1) + \left| \bigcup \operatorname{rng} F \restriction J \Big\backslash \bigcup \operatorname{rng} F \restriction K \right|.$$

Let $S \in P_\omega(K)$. There exists a finite, critical subfamily H^* of G^* such that $G^* \restriction S \subseteq H^*$. Let $T = \operatorname{dom} H^*$, then $S \subseteq T \subseteq K$ and

$$| \bigcup \operatorname{rng} F \restriction T | < (r + 1) + | T | ,$$

contrary to assumption.

We now present a generalization of a theorem of H. A. Jung and R. Rado.

THEOREM 7. *Let $F = (F(i) : i \in I)$ be a family, let $I_0 \subseteq I$, and let $F \restriction I\backslash I_0$ be a Hall-family. Then*

$$\exists g\, IA(F, g) \Leftrightarrow \exists f \Big(IA(F \restriction I_0, f)\ \&\ \forall J \in P_\omega(I\backslash I_0)$$
$$\times \Big(\Big|\bigcup \{F(j): j \in J\} \cap \operatorname{rng} f\Big| = k \Rightarrow \Big|\bigcup_{j\in J} F(j)\Big| \geqslant | J | + k\Big)\Big).$$

Proof. "$\Rightarrow$". Let g be an i.c.f. of F and $f = g \restriction I_0$. Then f is an i.c.f. of $F \restriction I_0$. If there is a finite subset J of I such that $k = | \bigcup_{j\in J} F(j) \cap \operatorname{rng} f |$ and $| \bigcup \{F(j) : j \in J\}| < | J | + k$, then f and therefore g are not injective functions. Contradiction!

"$\Leftarrow$". Let F be a family which satisfies the condition of Theorem 7, let f be an i.c.f. of $F \restriction I_0$, and let for $i \in I\backslash I_0$ $F^*(i) = F(i)\backslash \operatorname{rng} f$. If $J \in P_\omega(I\backslash I_0)$, then $| \bigcup_{j\in J} F^*(j)| \geqslant | J |$. By P. Hall's theorem and by Theorem 3, F^* possesses an i.c.f. Hence there exists an i.c.f. of the family F.

COROLLARY (Jung-Rado). *Let $F = (F(i) : i \in I^*)$ be a family, $i_0 \in I^*$, $I = I^*\backslash\{i_0\}$, and let $F \restriction I$ be a Hall-family. Then $\exists f\, IA(F, f) \Leftrightarrow \forall J \in P_\omega(I)$ $(| \bigcup \operatorname{rng} F \restriction J | \geqslant | J |)\ \&\ F(i_0) \subsetneqq \bigcup \{\operatorname{rng} G : G \subseteq F$ critical $\&$ G finite$\}$.*

Proof. We say that "F satisfies $\mathscr{H}$" if it is valid:

$$\forall J \in P_\omega(I)(|\bigcup \operatorname{rng} F \upharpoonright J| \geqslant |J|).$$

$\exists f\, IA(F, f)$

$$\Leftrightarrow \exists x \in F(i_0)\ \forall J \in P_\omega(I)$$

$$\times \left(\left|\bigcup \{F(j): j \in J\} \cap \{x\}\right| = k \Rightarrow \left|\bigcup \{F(j): j \in J\}\right| \geqslant |J| + k\right)$$

$$\Leftrightarrow F \text{ satisfies } \mathscr{H} \ \&\ \exists x \in F(i_0)\ \forall J \in P_\omega(I)$$

$$\times \left(x \in \bigcup \operatorname{rng} F \upharpoonright J \Rightarrow \left|\bigcup \operatorname{rng} F \upharpoonright J\right| \geqslant |J| + 1\right)$$

$$\Leftrightarrow F \text{ satisfies } \mathscr{H} \ \&\ \exists x \in F(i_0)\ \neg \exists J \in P_\omega(I)$$

$$\times \left(x \in \bigcup \operatorname{rng} F \upharpoonright J \ \&\ \left|\bigcup \operatorname{rng} F \upharpoonright J\right| = |J|\right)$$

$$\Leftrightarrow F \text{ satisfies } \mathscr{H} \ \&\ F(i_0) \nsubseteq \bigcup \{\operatorname{rng} F \upharpoonright J: J \in P_\omega(I) \ \&\ F \upharpoonright J \text{ critical}\}$$

$$\Leftrightarrow F \text{ satisfies } \mathscr{H} \ \&\ F(i_0) \nsubseteq \bigcup \{\operatorname{rng} G: G \subseteq F \text{ critical } \&\ G \text{ finite}\}.$$

Remark. (1) We do not know any criterion for families with only infinite members.

(2) The family $F = \{(\alpha, \omega): \alpha < \omega_1\}$ possesses neither a nonempty, critical subfamily nor an injective choice function. But we do not know if Theorem 4 can be extended to families with countably many infinite members. You will find further problems in [7].

Note Added in Proof. Recently question (2) was positively answered by K.-P. Podewski and the author.

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